Series Tests

Some Limits

$$\lim_{n \to \infty} r^n = \begin{cases} 0 & if & |r| < 1 \\ 1 & if & r = 1 \\ \infty & if & r > 1 \\ DNE & if & r \leq -1 \end{cases}$$

$$\lim_{n \to \infty} \sqrt[n]{n} = 1 \qquad\qquad \lim_{n \to \infty} \left(1 + \frac{a}{n}\right)^n = e^a$$

Specific Series Results

Geometric Series $$\sum_{n=1}^{\infty} ar^{n-1}$$

- Converges if $|r| < 1$
- Diverges otherwise

Specific Series Results

p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

- Converges if $p > 1$
- Diverges if $p \leq 1$

Specific Series Results

Harmonic Series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

Diverges

Specific Series Results

Alternating Harmonic Series $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{1}{n}$

Converges

The Divergence Test

7 **Test for Divergence** If $\lim_{n \to \infty} a_n$ does not exist or if $\lim_{n \to \infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

Result that goes along with this…

If $\lim_{n \to \infty} |a_n| \neq 0$, then $\lim_{n \to \infty} a_n \neq 0$

The Integral Test

The Integral Test Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\, dx$ is convergent. In other words:

(i) If $\int_1^{\infty} f(x)\, dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

(ii) If $\int_1^{\infty} f(x)\, dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

Comparison Tests

Direct

The Comparison Test Suppose that $\Sigma\, a_n$ and $\Sigma\, b_n$ are series with positive terms.

(i) If $\Sigma\, b_n$ is convergent and $a_n \leqslant b_n$ for all n, then $\Sigma\, a_n$ is also convergent.

(ii) If $\Sigma\, b_n$ is divergent and $a_n \geqslant b_n$ for all n, then $\Sigma\, a_n$ is also divergent.

The Limit Comparison Test Suppose that $\Sigma\, a_n$ and $\Sigma\, b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or both diverge.

Alternating Series Test

Alternating Series Test If the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots \qquad b_n > 0$$

satisfies

$$\text{(i)} \quad b_{n+1} \leqslant b_n \qquad \text{for all } n$$

$$\text{(ii)} \quad \lim_{n \to \infty} b_n = 0$$

then the series is convergent.

Ratio Test

The Ratio Test

(i) If $\lim\limits_{n \to \infty} \left| \dfrac{a_{n+1}}{a_n} \right| = L < 1$, then the series $\sum\limits_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(ii) If $\lim\limits_{n \to \infty} \left| \dfrac{a_{n+1}}{a_n} \right| = L > 1$ or $\lim\limits_{n \to \infty} \left| \dfrac{a_{n+1}}{a_n} \right| = \infty$, then the series $\sum\limits_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim\limits_{n \to \infty} \left| \dfrac{a_{n+1}}{a_n} \right| = 1$, the Ratio Test is inconclusive; that is, no conclusion can be drawn about the convergence or divergence of $\sum a_n$.

Root Test

The Root Test

(i) If $\lim_{n \to \infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(ii) If $\lim_{n \to \infty} \sqrt[n]{|a_n|} = L > 1$ or $\lim_{n \to \infty} \sqrt[n]{|a_n|} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n \to \infty} \sqrt[n]{|a_n|} = 1$, the Root Test is inconclusive.